CHAPTER 1

VECTOR CALCULUS

1.1 Introduction

Electrodynamics is the study of phenomena associated with charged bodies in motion and varying electric and magnetic fields. This area of electrodynamics was first systematically explained by James Clerk Maxwell. The four 'Maxwell's equations' together with the 'Lorentz force law' explain the entire electromagnetic interactions observed in nature[1]. In order to understand Maxwell's equations one must be familiar with the concepts of electric field, magnetic field, properties of charges, current etc. This chapter briefly describes the important points in vector calculus that one must be familiar with for the study of electrodynamics.

As you know, physics deals with the natural laws pertaining to physical quantities. Physical quantities, by definition, are measurable quantities. These are classified as scalars and vectors. (Nature did not create them as vectors and scalars. But, classifying physical quantities into scalars and vectors help us to understand the nature more succinctly. Scalars and vectors are subdivisions of a more general physical quantity called 'tensors'.) We define certain terms in physics whose entire physical relevance is understood by their numerical values. Such quantities are known as scalars. (Examples: mass, charge, etc.). But, certain physical quantities are defined such a way that their physical relevance is understood only when their magnitudes in different directions are specified. These are called vector quantities. For example, the effect of force is understood only when its magnitude and direction of action are specified. Therefore we recognise force as a vector quantity. Usually, in elementary treatments, a vector is defined as a quantity having magnitude and direction. In order to distinguish vectors from scalars, we identify vector quantities with bold letters in this book. For example, **V** is a vector and V is its magnitude. Displacement, electric field, magnetic field, angular momentum etc. are examples of vector quantities.

---

[1]Maxwell's equations and Lorentz force law are given in Appenix-1

1.2 Scalar Function

It is a function that assigns a real number (a scalar quantity) to each point in some region of space in which it is defined. Its value does not depend on the particular choice of the co-ordinate system.

The function ρ(x,y,z) which gives the charge density (i.e., charge per unit volume) at various points is an example of a scalar function. Similarly, the function T(x,y,z,t) which gives the temperature at various points in a region at a given time t is another example. To make it clearer, if the charge density is $\rho(x,y,z) = x^2+2y^2+z^2$, the charge per unit volume at (1,2,3) is 18.

1.3 Vector Function

If to each point on a certain set of points in space (for example, the points on a curve or a surface or a three dimensional region) a vector **V**(x,y,z) is assigned, **V** is called a vector function. In other words, a vector function is a function, which gives a vector quantity at all points in the domain in which it is defined.

Example:

1. If **V**(x,y) = y**i** – x**j** gives the velocity of flowing water, the velocity at points (1,1) and (2,2) are **i** – **j** and 2**i** – 2**j**, respectively. A plot of this vector function is shown in fig. 1.1a to get a physical idea of the water flower. Note that the length of the vector increases with distance from the origin. That is, in this case, water flows faster as the distance from the origin increases.

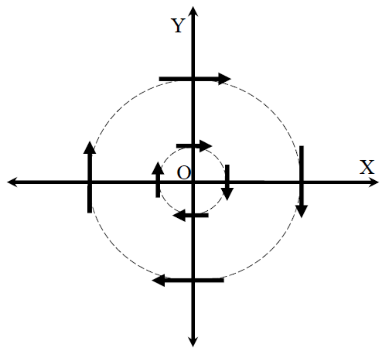


Fig.1.1a: The plot of the vector function y**i** – x**j**

2. The vector function given by **F**(r) = $-\frac{k}{r^2}\hat{r}$, where k is a constant, is plotted in fig.1.1b. (This vector function corresponds to the inverse square law of attractive force). It may be noted that the magnitude of the vector decreases with increase in distance from the origin. The direction of the vector field is towards the origin. Consider a charge −Q kept at the origin. The electrostatic field due to this charge at distance r is given by **E** = $-\frac{k}{r^2}\hat{r}$, where $k = \frac{Q}{4\pi\varepsilon_o}$.

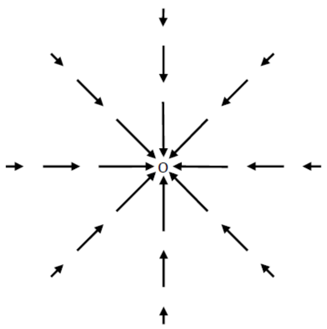


Fig.1.1b: The plot of the vector function $-\frac{k}{r^2}\hat{r}$

1.4 Vector Components

A vector may be conveniently represented by an arrow, with length proportional to its magnitude (see fig 1.2). The direction of the arrow gives the direction of the vector. In this representation, in order to add two vectors **A** and **B,** the rear of vector **B** is placed at the tip of vector **A**. The resultant vector **C** is then represented by the arrow drawn from the rear of **A** to the tip of **B**. This procedure is called the triangle law of vector addition. On the other hand, if we place the rear of **A** at the tip of **B**, we get **B**+**A**. It can be seen that **A**+**B** = **B**+**A**. Thus we see that the vector addition is commutative.

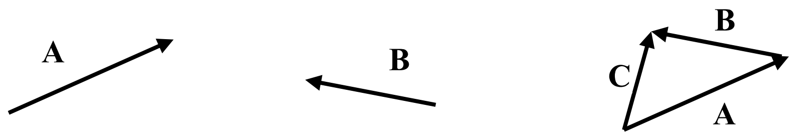


Fig. 1.2:
(a): Representation of **A** (b): Representation of **B** (c): **C** = **A**+**B**

Three orthogonal, i.e., mutually perpendicular, axes x, y and z are used in Cartesian co-ordinate system to specify a point. Let A be a point with co-ordinates (x,y,z). The vector drawn from the origin to A is called the **position vector** of A.

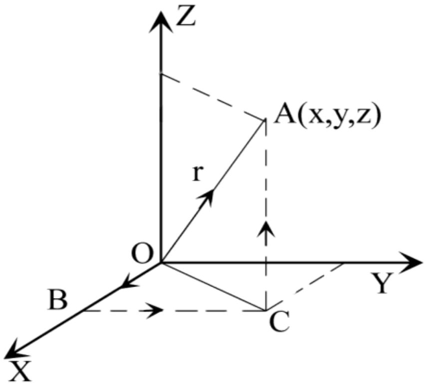


Fig.1.3: Components of a vector in Cartesian co-ordinate system.

A given vector can be resolved into its rectangular components along these x, y and z-axes as shown in fig.1.3. Let the vectors **i**, **j** and **k** represent unit vectors (unit vector is a vector with unit magnitude) along x, y and z axes respectively.

From fig.1.3, $\overrightarrow{OA} = \mathbf{r} = \overrightarrow{OB} + \overrightarrow{BC} + \overrightarrow{CA} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

Any arbitrary vector **A** can be represented as $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$. A_x, A_y and A_z are called the components of **A** along x, y and z axes. A_x, A_y and A_z are scalar quantities, which may be constants or scalar functions. The modulus or magnitude or length of **A** is $A = |\mathbf{A}| = (A_x + A_y + A_z)^{1/2}$. The vector **A** can be geometrically represented by a directed line segment with its length proportional to its modulus and an arrowhead showing its direction or by $A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}$ or by (A_x, A_y, A_z). The representation as (A_x, A_y, A_z) is called the component form. Two vectors are said to be equal if they have equal magnitude and direction, or each component of both the vectors are equal. The vector $\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j}$ lies in the x–y plane (since it does not have a component along z–axis). For example, the vectors given by the vector function $2y\mathbf{i} + 3xz\mathbf{k}$ are parallel to the x–z plane.

Let P be a point with coordinates(x_1,y_1,z_1) and Q be with coordinates (x_2,y_2,z_2) as shown in fig. 1.4. Then $\mathbf{r}_1 = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$ is the position vector (P.V.) of the point P and $\mathbf{r}_2 = x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$ is that of Q. The vector drawn to Q from P, $\mathbf{r}_{21} = \mathbf{r}_2 - \mathbf{r}_1$= the P.V. of the end point Q minus the P.V. of the initial point P.

Note: In this book we take the rear of the vectors as the origin, unless it is specified explicitly. Then the vector to any point (x,y,z) from the origin is just $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

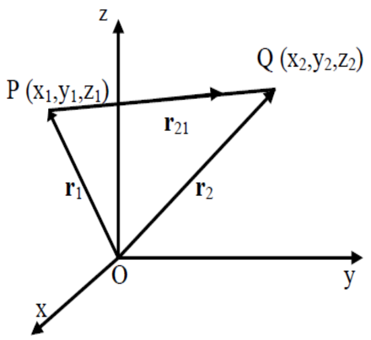


Fig.1.4: Vector drawn to Q from P,
$\mathbf{r}_{21} = \mathbf{r}_2 - \mathbf{r}_1$

If **C = A + B** and **C =** $\mathbf{C}_x\mathbf{i} + \mathbf{C}_y\mathbf{j} + \mathbf{C}_z\mathbf{k}$, then $C_x = A_x + B_x$, $\mathbf{C}_y = A_y + B_y$, and $\mathbf{C}_z = A_z + B_z$. It can also be proved that $C = (A^2+B^2+2AB\cos\theta)^{1/2}$, where θ is the angle between **A** and **B**.

Exercise:

1. If the displacement of a particle in a medium due to the propagation of a transverse wave along positive x–axis is **y**(x,t) = **A** sin (kx–ωt). What are the dimensional formulae of **y**, **A**, k and ω ?
2. The differential equation of a damped harmonic oscillator is $m\frac{d^2x}{dt^2} + C\frac{dx}{dt} + kx = 0$, where x is the displacement and m is the mass. What is the dimension of each term? What are the dimensional formulae of C and k?

1.5 Product of Two Vectors

One has to consider the directional properties of vectors while multiplying them. A simple multiplication of **A** with **B**, like scalars, is meaningless and not defined, i.e., (**A**)(**B**) is '*not'* defined. [The dyadic product **AB** is defined, but it is beyond the scope of this book]. While multiplying two vectors, one has to take either a dot product or a cross product as explained below.

1.5.1 Dot Product or Scalar Product or Inner Product

The dot product of **A** with **B** is **defined** as **A.B** = |**A**| |**B**| cosθ, where θ is the angle between **A** and **B**. Note that the dot product of two vectors is a scalar quantity.

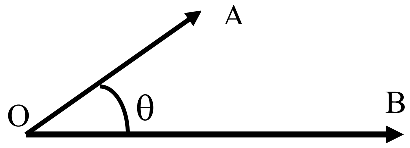


Fig.1.5: Two vectors **A** and **B** with an angle θ between them.

We get, from the definition, **i.i** =|**i**| |**i**|cos 0 = 1×1×1= 1 and **i.j** = 1×1×cos 90 = 0. We get similar results for the other combinations. That is,

i . i = 1	**i . j** = 0	**i . k** = 0
j . i = 0	**j . j** = 1	**j . k** = 0
k .i = 0	**k. j** = 0	**k .k** = 1

The dot product is commutative: **A.B** = **B.A**

The dot product is distributive: **A.**(**B** + **C**) = **A.B** + **A.C**

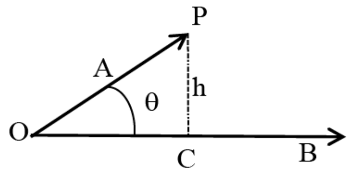


Fig.1.6: Figure to show the geometrical significance of dot product.

From fig.1.6, $\cos\theta = \frac{OC}{OP}$. Therefore, OP cosθ = OC = projection of **A** on **B**.

A.B = |**A**| |**B**| cosθ = |**B**||**A**| cosθ = (B)(Acosθ) = (B)$|\overrightarrow{OC}|$. Similarly, it can also be proved that **A.B** is the product of the modulus of **A** and the projection of **B** on **A,** i.e., **A.B** = **B.A.** The geometrical significance of **A.B** is that it is equal to the product of the length of one vector and the projection of the other on the first vector.

If **A** = A_x**i** + A_y**j** + A_z**k** and **B** = B_x**i** + B_y**j** + B_z**k**, then **A.B** = $A_xB_x + A_yB_y + A_zB_z$.

An example to show the application of dot product: Let **F** = (3**i** + 4**k**) newton be the force applied on an object to cause a displacement **S** = (2**i** + 4**j** + 5**k**) metre. The work done is then given by W = **F.S** = 6 + 0 + 20 = 26 joule.

1.5.2 Cross Product or Vector Product

The cross product of two vectors **A** and **B** is **defined** as **A**×**B** = |**A**| |**B**| sinθ$\hat{n}$, where θ is the angle between **A** and **B** and $\hat{n}$ is the unit vector perpendicular to both **A** and **B**. The cross product of two vectors is a vector, which is perpendicular to the plane defined by **A** and **B**. The direction of $\hat{n}$ is given by the right-handed screw rule: If a right-handed screw is rotated from the direction of **A** to that of **B**, the advancing end of the screw gives the direction of $\hat{n}$. Or, hold the right hand fingers as shown in fig. 1.7, with the fingers showing the direction of rotation from **A** to **B**. Then the outward stretched thumb gives the direction of cross product.

The direction of unit vector $\hat{n}$ changes sign when the order of multiplication is reversed, i.e., $(\mathbf{A} \times \mathbf{B}) = -(\mathbf{B} \times \mathbf{A})$. Therefore the cross product is not commutative.

The cross product is distributive: $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$

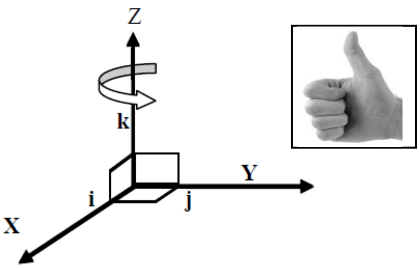


Fig.1.7: The right-handed orthogonal Cartesian co-ordinate system. Figure also shows $\mathbf{i}\times\mathbf{j} = \mathbf{k}$

Since the angles between the unit vectors **i**, **j** and **k** are 90^0 (see fig.1.7), $\mathbf{i}\times\mathbf{i} = (|\mathbf{i}|)(|\mathbf{i}|)(\sin 0)\hat{n} = (1)(1)(0)\hat{n} = \mathbf{0}$ (**0** is a null vector; a null vector is a vector with zero magnitutde) and $\mathbf{i}\times\mathbf{j} = (|\mathbf{i}|)(|\mathbf{j}|)(\sin 90)\hat{n} = (1)(1)(1)\hat{n}$. But the unit vector perpendicular to both **i** and **j** and that obeys right-handed screw rule is **k**. (When a right-handed screw is rotated from the direction of x–axis to the direction of y–axis, it advances along positive z–axis, see fig.1.7.) Thus, we get $\mathbf{i}\times\mathbf{j} = \mathbf{k}$. In general,

$$\mathbf{i}\times\mathbf{i} = \mathbf{0} \qquad \mathbf{i}\times\mathbf{j} = \mathbf{k} \qquad \mathbf{i}\times\mathbf{k} = -\mathbf{j}$$

$$\mathbf{j}\times\mathbf{i} = -\mathbf{k} \qquad \mathbf{j}\times\mathbf{j} = \mathbf{0} \qquad \mathbf{j}\times\mathbf{k} = \mathbf{i}$$

$$\mathbf{k}\times\mathbf{i} = \mathbf{j} \qquad \mathbf{k}\times\mathbf{j} = -\mathbf{i} \qquad \mathbf{k}\times\mathbf{k} = \mathbf{0}$$

Let $\mathbf{C} = \mathbf{A}\times\mathbf{B}$. Then,

$$\mathbf{C_x i} + \mathbf{C_y j} + \mathbf{C_z k} = (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) \times (B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k})$$

$$= (A_yB_z - A_zB_y)\,\mathbf{i} + (A_zB_x - A_xB_z)\,\mathbf{j} + (A_xB_y - A_yB_x)\,\mathbf{k}$$

Equating the coefficients, gives $C_x = (A_yB_z - A_zB_y)$. The other coefficients C_y and C_z are obtained by changing x, y and z cyclically. Or, the cross product can be found by

$$\mathbf{C} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Geometrical Significance of Cross-Product:

The fig.1.8 shows a parallelogram with vectors **A** and **B** as adjacent sides. The area of the parallelogram = |**B**|×h. From fig.1.8, h = |**A**| sin θ.

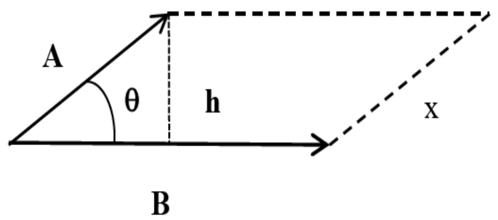


Fig.1.8: Figure to show the geometrical significance of vector cross product.

Therefore, the area = (|**B**|)(|**A**| sinθ) = |**A**×**B**|. That is, the modulus of **A**×**B** gives the area of the parallelogram of which **A** and **B** are adjacent sides. The vector **A**×**B** is perpendicular to both **A** and **B** and, in this case, it is pointing into the plane of the paper. If we represent the area of this parallelogram by a vector **S** = **A**×**B.** The modulus of **S** is equal to the geometrical area of the parallelogram. The vector **S** is perpendicular to the plane defined by **A** and **B**. Area is considered as a vector and the **area vector** is perpendicular to the plane of the area element. By convention, if the surface is a closed one, **the outward normal** on the surface **is taken as the positive direction** of the area vector.

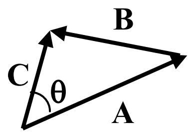
An example for the vector cross product in physics: The angular momentum **L** about the origin of a particle with position vector **r** and linear momentum **p** is, **L** = **r**×**p**.

Exercise:

3. Let **A** = 3**i** + 4y**j** + 8x^2**k** and **B** = y**i** + 3**j** – **k**
   (a) Find **A**+**B** and **A**–**B**
   (b) Find **A.B** and **A** × **B**.

4. Find the angles between **A** = 2**i** – 4**j** + 5**k** and x, y and z-axes?
5. Two forces 3**i** – 5**k** N and 3**k** – 4**j** N act on a particle simultaneously. Find the magnitude and direction of the resultant force.
6. Determine the angle between the vectors **A**= **i** – 2**j** + 3**k** and **B** = 3**i** – 2**j** + 4**k**.
7. Show that 7**i**–5**k** is perpendicular to 3**j**.
8. According to the definition of work in physics, a man walking on a level road carrying a bag is not doing any work. Why?
9. Find the area of the parallelogram of which **A** = 3**i** – 4**j** and **B** = 4**j** – 3**k** are adjacent sides.
10. Prove the law of cosines of a triangle.

(i.e., Show that $B^2 = A^2 + C^2 - 2AC\cos\theta$ for the triangle shown.)



1.6 Triple Products

Since **B**×**C** is a vector, one can take the dot product of this vector with another vector **A** resulting a scalar. This is known as **scalar triple product** or **box product**. Similarly, one can take the cross product of this vector **B**×**C** with another vector **A** resulting a vector. This is known as **vector triple product**.

1.6.1 Scalar Triple Products

The scalar triple product of three vectors is defined as **A**.(**B**×**C**). It can be shown that **A**.(**B**×**C**) gives the volume of the parallelepiped with **A**,**B** and **C** are sides with common rear point (see fig. 1.9)

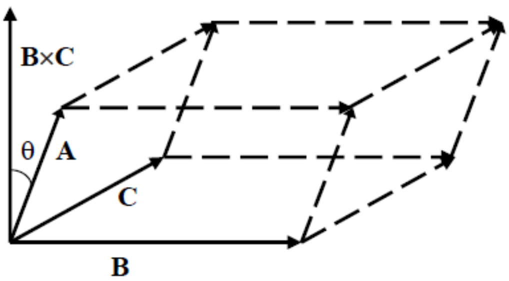


Fig.1.9: **A.**(**B**×**C**) gives the volume of the parallelepiped shown in the figure.

The scalar triple product remains the same when the vectors are changed cyclically, i.e., $\mathbf{A}.(\mathbf{B}\times\mathbf{C}) = \mathbf{B}.(\mathbf{C}\times\mathbf{A}) = \mathbf{C}.(\mathbf{A}\times\mathbf{B})$. Even though $\mathbf{A}.(\mathbf{B}\times\mathbf{C})$ is a scalar, it is different from the ordinary scalar quantities. The cross product changes sign when the order of multiplication is reversed, i.e., $\mathbf{A}.(\mathbf{B}\times\mathbf{C}) = -\mathbf{A}.(\mathbf{C}\times\mathbf{B})$. Hence the scalar triple product changes sign when the reference axes are interchanged. Such scalars whose sign depends on the orientation of the reference axes are called **pseudo scalars**.

When the vectors **A**, **B** and **C** are represented in the component form, $\mathbf{A}.(\mathbf{B}\times\mathbf{C})$ is obtained by the determinant of the components.

i.e.,
$$\mathbf{A}.(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Note: The vectors obtained as the cross product of two vectors are called **pseudo vectors or axial vectors**. For example, the angular momentum vector $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ is a pseudo vector. The scalar product of a pseudo vector and an ordinary vector is a pseudo scalar. A simple explanation of axial and polar vectors are given below.

Axial vector: (1) The axial vectors are used to express the rotational effect, (2) The axial vector directed towards the axis of rotation. Examples: Torque, Angular velocity, and Angular acceleration. **Polar Vector:** The Polar vector is used to express the directed properties. Examples of the polar vector are displacement and force.

Note: While writing any equation in physics, if the left-hand side is a scalar (or a vector), the right-hand side also must be a scalar (or a vector). More precisely, if $\mathbf{A} = \mathbf{B}$, both **A** and **B** are either axial vectors or a polar vectors.

1.6.2 Vector Triple Products

Consider the vector obtained by $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$. The vectors **B** and **C** define a plane. The vector $\mathbf{B}\times\mathbf{C}$ is perpendicular to both **B** and **C**. Hence, $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ lies in the plane perpendicular to both **A** and $(\mathbf{B}\times\mathbf{C})$. That is, $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ lies in the same plane defined by **B** and **C**. Therefore, $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ can be represented as the linear combination of **B** and **C**. On expanding $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ using the component form it can be shown that

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}.\mathbf{C}) - \mathbf{C}(\mathbf{A}.\mathbf{B}).$$ (This is referred to as **B**AC-**C**AB rule.)

1.7 CALCULUS

1.7.1 Differentiation

Let y be a function of x: y = f(x). Here x is the independent variable and y is the dependent variable. The derivative of y with respect to x (written as $\frac{dy}{dx}$) is defined as the ratio of the change Δy in y due to a change Δx in x, when Δx is infinitesimally small.

That is, $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$.

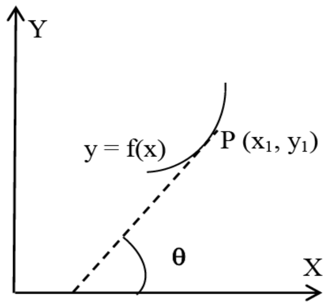


Fig.1.10: The derivative of y at a point P is the slope of the curve at that point. In the figure, $\frac{dy}{dx}$ at (x_1,y_1) is equal to $\tan\theta$.

If we plot a graph between x and y = f(x), the derivative of y with respect to x at any point (x_1,y_1) gives the slope of the curve y(x) at that point.

The idea of finding the derivative and its advantage can be understood from the following simple example. The displacement **S** of a freely falling object from rest is $\mathbf{S}(t) = \mathbf{S}_0 + (\frac{1}{2})\mathbf{g}t^2$.

The velocity **v**, which is the rate of displacement, is

$$\mathbf{v} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{S}}{\Delta x} = \frac{d\mathbf{S}}{dt} = \mathbf{g}t$$

The acceleration **a**, which is the rate of change of velocity, is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(\frac{d\mathbf{S}}{dt}\right) = \frac{d^2\mathbf{S}}{dt^2} = \mathbf{g}$$

The idea of differentiation can be extended further by taking the directional properties of the derivatives into account. Among these, the simplest and important ones that we use in this book are the gradient of a scalar functions (section 1.9) and the divergence (section 1.10) and curl (section 1.11) of vector functions.

1.7.2 Integration

If $\frac{d}{dx}[F(x)] = f(x)$ we say that F(x) is an integral or primitive of f(x). In symbols, we write $\int f(x)dx = F(x)$, and f(x) is called the integrand. Thus, the integral of a function is that function which on differentiation yields the original function as its derivative.

The process of integration is equivalent to continuous summation. For example, consider a charged wire of length L with charge per unit length, called linear charge density, λ. Assume that the wire is divided into n pieces, each of length dl. Then the charge dq on a small line segment dl is λdl (fig. 1.11). Therefore, the total charge on the wire segment $Q = \sum_{i=1}^{n} dq_i = \sum_{i=1}^{n} \lambda_i dl_i$. When the number of pieces tends to infinity, i.e., when dl becomes negligibly small, $Q = \sum_{i=1, n\to\infty}^{n} \lambda_i dl_i$ and, this is equivalent to $\int_o^L \lambda dl$.

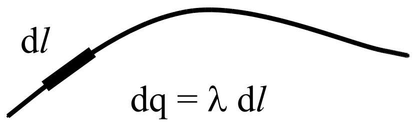


Fig.1.11: A charged wire of length L with charge per unit length λ.

The idea of integration can be made clearer using the following example.

Consider a non-uniform distribution of charge on a wire. Let the linear charge density λ be proportional to the distance from one end, i.e., $\lambda \propto x$ or $\lambda = k\,x$, where k is a constant and x is the distance from the end. Here, since the charge distribution is continuous, the total charge up to any distance (say up to x = 10) is calculated using integration. (Here, in this example, note that the charge density is a function of x.)

The total charge up to x = 10 is $\int_o^{10} \lambda dx = \int_o^{10} (kx)dx = \left[\frac{kx^2}{2}\right]_0^{10} = k\times 50$ (1)

If k =1, we get λ = 0 at x = 0, λ =1 at x =1, λ =2 at x = 2, and so on. This is shown in fig.1.12. Eqn. (1) gives Q = 50 if k = 1. That is, the total charge up to x = 10 is 50.

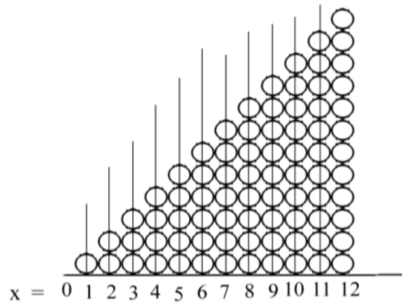


Fig.1.12: An example to show that the integration is equivalent to continuous summation. Plot of $\lambda = kx$ with k =1. [See text.] The charge within the intervals x = 0 and x =1 is 0.5, x =1 and x = 2 is 1.5, x = 2 and x = 3 is 2.5, ……., and x = 9 and x =10 is 9.5. Thus the total number of charges up to x = 10 is 0.5+1.5 +2.5+ 3.5+ ……..+9.5 = 50.

In a similar way, the total charges on a surface is written as $\int_{surface} \sigma da$, and inside a volume as $\int_{volume} \rho dv$, where σ is the surface charge density (i.e., charge per unit area), and ρ is the volume charge density (i.e., charge per unit volume). Later we will use the symbol '$\oint$' to denote closed integrals, i.e., to specify that the integration is done over a closed loop or a closed surface.

In general, the integrand can be either a scalar or a vector. Till now, we were discussing about the integrations (line, surface or volume integration) of scalar quantities (λ, σ and ρ described above are scalar quantities). On the other hand, if the integrand is a vector, it is called vector integration. Among the various types of vector integrations, we are interested only in the scalar line integral of vectors, scalar surface integral of vectors and volume integral of vectors. These three types are represented as $\int_A^B \mathbf{A}.d\boldsymbol{l}$, $\int_{surface} \mathbf{A}.\,d\mathbf{s}$ and $\int_{volume} \mathbf{A}dv$, respectively, and we discuss about these in section 1.8.

1.7.3 Integration by parts

We have the general methods, such as chain rule, functions of function rule, quotient rule, product rule etc., for finding the derivative of functions. But, unfortunately, there is no such general method available for integration. One of the most powerful tools used in integration is the integration by parts rule. It is stated as follows: The integral of the product of two functions = (product of the first function and the integral of the second) − (the integral of the product of the derivative of the first and the integral of the second).

$$\int (uv)dx = u\int vdx - \int \left[\frac{du}{dx}\int vdx\right]dx$$

Or $$\int (u\frac{dv}{dx})dx = uv - \int \left[\frac{du}{dx} v\right]dx$$

Note: Suppose either the function u or v (or both) vanishes at the boundaries a and b. Then, $\int_a^b (u\frac{dv}{dx})dx = uv\Big|_a^b - \int_a^b \left[\frac{du}{dx} v\right]dx = -\int_a^b \left[\frac{du}{dx} v\right]dx$. That is, the differential operation can be shifted from v to u. This idea is widely used in vector calculus.

Exercise:

11. If the displacement of a particle is given by **S** = $(7t-9t^2+3)$**k**, find the velocity and acceleration of the particle when t = 3 s.
12. The motion of a particle is given by parametric equations x = 4sin2t, y = 4cos2t, z = 6t. Find the velocity and acceleration of the particle if the co-ordinates are expressed in meters.

13* A current I flows through a wire of radius R.

(a) If the current is uniformly distributed over the surface, what is the surface current density (current through unit length perpendicular to the direction of the flow of charges)?

(b) If it is distributed in such a way that the volume current density is inversely proportional to the distance from the axis, what is the volume current density (current through unit area)? [The term current density generally refers to the volume current density, unless it is specified.]

14* If the volume charge density of a charged sphere of radius 6 m is proportional to the distance from the centre as ρ =3r, find the total charge on the sphere. What is the charge density at the radius equal to 2 m?

15. The displacement of a particle is **r** = $[10t^3-5t^2)$**i** + $5t^2$**j** + (t^2-5)**k**] m. Determine the angular momentum about the origin of the co-ordinate system and the torque on it at t =1s. [Mass of the particle = 0.010 kg.]

1.8 Three Important Integrals

(1) Scalar Line Integral of Vector Functions

This is one of the simplest and most useful vector integration. Suppose we want to move a particle from point A to B by applying a force. Divide the path from A to B into n number of small segments each having length d*l*. We know the work done by a force field **F** in displacing a particle through infinitesimal displacement 'd***l***' is dw= (F)(d*l*)(cosθ) = **F**.d***l***. If the force is a variable force or if the angle between force and the displacement vector is not a constant, 'dw' will be different for different line segments. The total work done in taking the particle from point A to point B is $W=\sum_{i=1,}^{n} \mathbf{F}_i . d\boldsymbol{l}_i$. In the limit n→∞, the summation becomes integration along the path from A to B, and we write $W = \int_A^B \mathbf{F}.d\boldsymbol{l}$. This is the scalar line integral of a vector function (or vector field) **F** along the curve from A to B.

It is important to note that this line integral usually depends on the path along which the integration is done and also on the end points A and B. Further, the integral over a closed path may or may not be zero, i.e., $\oint_{line} \mathbf{F}.d\boldsymbol{l}$ need not be zero, in general. However, there are many situations where $\oint_{line} \mathbf{F}.d\boldsymbol{l}$ = 0, and in such cases we call the vector field **F** a conservative field.

(2) Scalar Surface Integral of Vector Functions

Flux is a property of a vector field and is defined as follows:

Flux = (average normal component of the vector field)**.**(surface area)

Consider a surface S as shown in fig. 1.13. Divide the surface into small sections ds_1, ds_2, ... etc. with representative vectors $d\mathbf{s}_1$, $d\mathbf{s}_2$, ... etc. Let $\mathbf{V}_i$ be the value of the vector function $\mathbf{V}(x,y,z)$ at $d\mathbf{s}_i$. Then, $\lim_{ds_i \to 0} \sum_{i=1, n\to\infty}^{n} \mathbf{V}_i . d\mathbf{s}_i = \int_{surface} \mathbf{V}.d\mathbf{s}$ is called the scalar surface integral of **V** over the surface S or the flux of **V** over the surface.

[$\mathbf{V}.d\mathbf{s} = V.\hat{n}\, d\mathbf{s}$ where $\hat{n}$ is the unit vector perpendicular the surface d**s**.

$\mathbf{V}.\hat{n} = V\cos\theta$ = the component of **V** perpendicular to the surface element d**s**.]

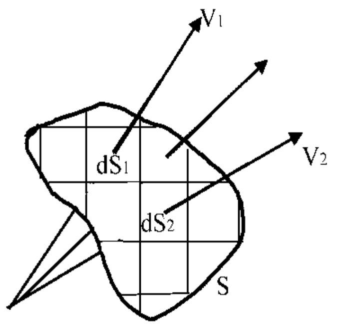


Fig.1.13: A vector field **V** passing through a surface S.

If the surface is a closed one, the outward normal is taken as the positive direction. Since d**s** is a vector, we write $d\mathbf{s} = ds_x\mathbf{i} + ds_y\mathbf{j} + ds_z\mathbf{k}$. Then,

$$\int_{surface} \mathbf{V}.d\mathbf{s} = \int_{surface} (V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k}) \cdot (ds_x\mathbf{i} + ds_y\mathbf{j} + ds_z\mathbf{k}) = \int_{surface} (V_x ds_x + V_y ds_y + V_z ds_z)$$

The idea of surface integral is explained using the following example.

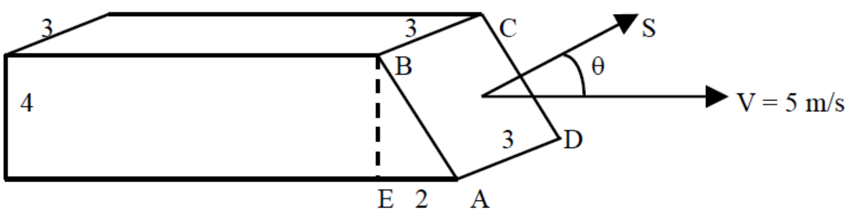


Fig.1.14: A rectangular tube through which water flows with velocity 5 m/s towards right.

Consider a water flow through a rectangular tube with velocity 5m/s along x−direction. Let the height and breadth of the tube be 4 and 3 meters, respectively (see the figure 1.14). Let the tube be cut at one end such that AE = 2m, EB = 4m, BC = 3 m and AD = 3m.

We want to get the total quantity of water flowing out of this tube per second. This problem can be solved in two ways.

1. The area of cross section of the tube is 4 × 3 = 12 m^2. The velocity is 5 m/s. Therefore the quantity of water flowing out per second is given by (5×12) = 60 m^3.
2. Let the point A be the origin. Also let $\overrightarrow{AD}$ be along y-direction, $\overrightarrow{AE}$ along negative x−axis and $\overrightarrow{EB}$ along z−direction. Therefore the vector $\overrightarrow{AD}$ is 3**j**, $\overrightarrow{AE}$ is −2**i** and $\overrightarrow{EB}$ is 4**k**. The velocity vector **V** = 5**i**. Area of the face ABCD is $\overrightarrow{AD} \times \overrightarrow{AB}$ (refer to section 1.2.2 to understand how the vector cross product is used to represent area).

Area **S** = $\overrightarrow{AD} \times \overrightarrow{AB}$ = 3**j**×(−2**i** + 4**k**) = 6**k** + 12**i**.

The component of **V** along the direction of **S,** i.e., perpendicular to the surface ABCD is Vcosθ, where θ is the angle between **V** and **S**. Therefore the total quantity of water flowing per second perpendicular to the surface is (V cosθ)S = **V.S**.

$$\mathbf{V.S} = (5\mathbf{i}).(6\mathbf{k} + 12\mathbf{i}) = 60 \text{ m}^3/\text{s}$$

Or, the component of **S** along **V** is S(cosθ). Therefore, the quantity of water that flows along x−axis is V(S cosθ) = **V.S** = 60 m^3/s.

This answer '60' should be independent of the shape of the end surface. If the end surface is not flat, we divide the end surface into a large number of infinitesimally small surface elements and evaluate $\lim_{ds_i \to 0} \sum_{i=1,n\to\infty}^{n} \mathbf{V}_i.d\mathbf{s}_i$ (i.e., $\int_{surface} \mathbf{V.ds}$) to get the net flow per second. When we evaluate $\int_{surface} \mathbf{V.ds}$, the answer will be 60 m^3/s. This is the physical meaning of scalar surface integral of vector functions.

Suppose we want to find out the volume of a liquid flowing out per second from a closed surface. It is calculated as $\oint_{surface} \mathbf{V.ds}$, where $\mathbf{V}$(x,y,z) is the vector function that gives the velocity of fluid inside the volume. If $\oint_{surface} \mathbf{V.ds}$ comes out to be negative, it simply means that the liquid is entering into the volume bounded by the surface.

(3) Volume Integral of Vector Functions

We know from section 1.7 that the net charge on a material is Q = $\int_{volume} \rho\, dv$, where ρ is the charge density function. Similarly, one can do $\mathbf{G} = \int_{volume} \mathbf{J}\, dv$ for a vector field. This expression contains three equations in three dimensional space. For example, the x–component of **G** is given by $G_x = \int_{volume} J_x\, dv$. We will come across this type of integral when we deal with vector potentials in magnetism.

1.9 Gradient

If f_1 is the value of a scalar function f(x,y,z) at (x_1,y_1,z_1), then f_2, the value at $(x_1+\Delta x, y_1, z_1)$, is $f_2 = f_1 + \frac{\partial f}{\partial x}\Big|_{at(x_1,y_1,z_1)} \times \Delta x$

Here it is assumed that Δx is infinitesimally small. Similarly f_3, the value of the function at $(x_1+\Delta x, y_1+\Delta y, z_1+\Delta z)$, is

$$f_3 = f_1 + \frac{\partial f}{\partial x}\Big|_{at(x_1,y_1,z_1)} \times \Delta x + \frac{\partial f}{\partial y}\Big|_{at(x_1,y_1,z_1)} \times \Delta y + \frac{\partial f}{\partial z}\Big|_{at(x_1,y_1,z_1)} \times \Delta z$$

The change $\Delta f = f_3 - f_1 = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \frac{\partial f}{\partial z}\Delta z$

$$\Delta f = (\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}).(\Delta x\,\mathbf{i} + \Delta y\,\mathbf{j} + \Delta z\,\mathbf{k}) = \nabla f \,.\, \Delta \boldsymbol{l} \qquad (1)$$

Here '$\Delta \boldsymbol{l}$' stands for the infinitesimal displacement vector Δx **i** + Δy **j** + Δz **k** and the '∇' symbol (called the grad or gradient or del operator) stands for $\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$.

∇f is a <u>vector function</u>, called the gradient of f(x,y,z).

Eqn.(1) needs some explanation. For a given f(x,y,z), we get the vector function ∇f as $\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}$. This $\nabla f(x,y,z)$ has a specific direction and magnitude at a given point. $d\boldsymbol{l}$ is the infinitesimal displacement given by $\Delta x\ \mathbf{i} + \Delta y\ \mathbf{j} + \Delta z\ \mathbf{k}$ from the point (x,y,z). Δf becomes zero if ∇f and $d\boldsymbol{l}$ are perpendicular. The surface over which the function 'f' has a constant value is called a **level surface**. (If the scalar function represents the potential, we call this surface an **equipotential surface**.) That is, if $d\boldsymbol{l}$ lies on a level surface Δf is zero. Since $|\nabla f|$ and $|d\boldsymbol{l}|$ cannot be zero in general, ∇f has to be perpendicular to $d\boldsymbol{l}$. That is, the vector <u>∇f is perpendicular to a level surface.</u>

If $\hat{x}$ is the unit vector along $\Delta\boldsymbol{l}$, $\Delta f = \nabla f.\Delta\boldsymbol{l} = \nabla f.\hat{x}\ |\Delta\boldsymbol{l}| = (|\nabla\mathbf{f}|\cos\theta)\ |\Delta\boldsymbol{l}|$ (where θ is the angle between $\nabla\mathbf{f}$ and $\Delta\boldsymbol{l}$). **In words:** Δf = The product of the projection of ∇f along the direction of $\Delta\boldsymbol{l}$ and the length of the vector $\Delta\boldsymbol{l}$.

If $|\Delta\boldsymbol{l}|=1$, $\Delta f = \nabla f.\hat{x}$. That is, <u>the change in f along any direction for unit displacement is given by $\nabla f.\hat{x}$, where $\hat{x}$ is the unit vector along that direction</u>.

From vector dot product, $\nabla f.\Delta\boldsymbol{l} = |\nabla f|\ |\Delta\boldsymbol{l}|\cos\theta$ (2)

(where θ is the angle between ∇f and $\Delta\boldsymbol{l}$.)

From eqns. (1) and (2), $\frac{\Delta f}{|\Delta\boldsymbol{l}|} = |\nabla f|\cos\theta$. The value of $\frac{\Delta f}{|\Delta\boldsymbol{l}|}$ changes depending on the angle θ between the vectors ∇f and $\Delta\boldsymbol{l}$. Thus, in the limit $\Delta\boldsymbol{l} \to 0$, we get

$$\lim_{|\Delta\boldsymbol{l}|\to 0} \frac{\Delta f}{|\Delta\boldsymbol{l}|} = \frac{df}{dl} = |\nabla f|\cos\theta$$

or the change, $df = |\nabla f|\cos\theta\ (dl)$ (3)

From eqn(3), we see that that the change df in f for a given dl depends on the angle between ∇f and $d\boldsymbol{l}$. If ∇f and $d\boldsymbol{l}$ are in the same direction we get the maximum value of df. Then, $\frac{df}{dl} = |\nabla f|$. In other words, <u>the modulus of the vector function ∇f at a point is equal to the maximum value of $\frac{df}{dl}$, and points in the direction in which the change in f is maximum for infinitesimal change in special variables.</u>

Consider a gas-filled non-conducting vessel with bottom surface kept at 0^{o}C and the top surface at 100^{o}C. In this case, the density of the gas is maximum at the bottom and minimum at the top. If D(x,y,z) is the scalar function which gives the density of the gas inside the vessel, then ∇D is a vector function that points downward. That is, as one moves from a point (x,y,z), he/she will find that the rate of increase in density of the gas is maximum when moves downward. The magnitude of ∇D gives the change ΔD for unit displacement in the downward direction. Similarly, if T(x,y,z) is the function which gives the temperature of the gas inside the vessel, then ∇T points upward. It means that as one moves from any point (x,y,z), he/she will find that the rate of increase of temperature is maximum when moves upward.

Exercise:

16* Show that $\nabla \frac{1}{r} = -\frac{\hat{r}}{r^2}$ where r is the modulus of **r** given by x**i** + **yj** + z**k**.

17* Verify $\nabla' \frac{1}{R} = \frac{\hat{R}}{R^2}$ where $\nabla' = \mathbf{i}\frac{\partial}{\partial x'} + \mathbf{j}\frac{\partial}{\partial y'} + \mathbf{k}\frac{\partial}{\partial z'}$, **R** = **r** – **r′**, **r** = x**i** + **yj** + z**k** and **r′**= x′**i** + y′**j** + z′**k**.

18* Show that $\nabla r = \hat{r}$.

1.10 Divergence

This is a type of differentiation done on vector functions. The divergence of a vector field at a point gives the measure of the vector field spreading out from the point considered. If τ is the infinitesimal volume enclosed by an infinitesimally small closed surface surrounding a point P(x,y,z), then the divergence of **V** at P is defined as,

$$\text{div V (written as } \nabla.\mathbf{V}) = \lim_{\tau \to 0} \frac{\oint_{surface} \mathbf{V.ds}}{\tau}$$

We know that $\oint_{surface} \mathbf{V.ds}$ is the net flux over the closed surface.

Thus, the divergence of **V** is the net flux from unit volume. The divergence of **V**(x,y,z) in Cartesian system is,

$$\nabla.\mathbf{V} = (\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}).(V_x\mathbf{i} + V_y\mathbf{j} + V_z\mathbf{k}) = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

[Note that the divergence of a vector function is a scalar function. The divergence itself is an operator that gives certain meaningful physical idea when it operates on vector functions.]

Let **F**(x,y,z) be a vector function defined in a region. If this vector function converges at certain points, we say, its divergence is negative at these points, and if the function is spreading out from certain points, its divergence is positive at these points. (See the fig.1.15)

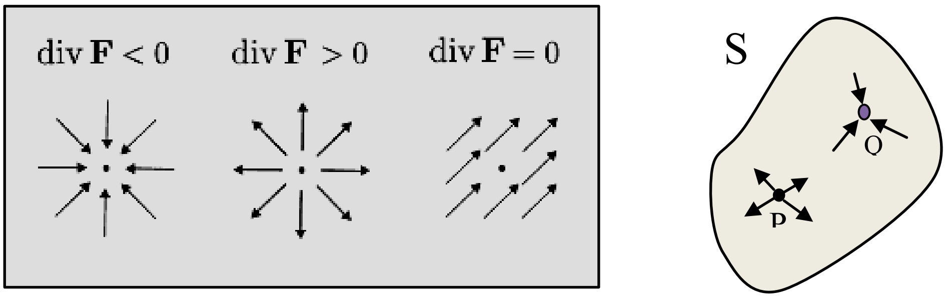


Fig.1.15: A hypothetical surface S enclosing some charges. At the point P, the net charge is positive and at Q it is negative. The function **E** representing the electric field in this enclosed surface has positive divergence at P and negative divergence (or convergence) at Q.

Consider an imaginary surface enclosing some charges. Also assume that the net charge at P is positive and that at Q is negative (see fig. 1.15). If **E** represents the electric field inside this enclosed surface, $\nabla.\mathbf{E}$ is positive at P and negative at Q. That is, **E** spreads out from P and sinks in at Q. In other words, the point P acts as a **source** of **E** and Q as a **sink**. For all other points in the region $\nabla.\mathbf{E} = 0$. A vector with zero divergence is called a **solenoidal** vector.

Exercise:

19. Show that $\nabla.\ [(y^2 - z^2)\mathbf{i} + 3x^2\mathbf{j} + (x^2 + y^2)\,\mathbf{k}] = 0$

1.11 Curl (or Circulation or Rotation)

The curl of a vector function **B**(x,y,z) = B_x**i** + B_y**j** + B_z**k** is calculated in Cartesian coordinates as

$$\nabla\times\mathbf{B} = (\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}) \times (B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k})$$

$$= (\frac{\partial}{\partial y}B_z - \frac{\partial}{\partial z}B_y)\,\mathbf{i} + (\frac{\partial}{\partial z}B_x - \frac{\partial}{\partial x}B_z)\mathbf{j} + (\frac{\partial}{\partial x}B_y - \frac{\partial}{\partial y}B_x)\mathbf{k}$$

i.e., $$\nabla\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}$$

Curl of a vector function is also a vector function. Physically, the curl of a vector function at a point gives how much of the vector quantity curls around the point considered. For example, let us assume that **V** = y**i** – x**j** gives the velocity of water flow in certain region. Then

$$\nabla\cdot\mathbf{V} = (\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z})\cdot(y\,\mathbf{i} - x\,\mathbf{j}) = 0$$

This shows that water neither accumulates at nor diverges from any point.

$$\nabla\times\mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{dz} \\ y & -x & 0 \end{vmatrix} = (\frac{\partial}{\partial x}(-x) - \frac{\partial}{\partial y}y)\mathbf{k} = -2\mathbf{k}$$

That is, the curl of **V** is along negative z–axis. This shows that water curls around the z–axis. Notice that the vector function **V** is in the x–y plane and the curl of **V** is along the negative z–axis.

Let us examine the physical significance of divergence and curl using the representation of the vector function **V**. Fig. 1.16 shows the plot **V** = y**i** – x**j.** It can be seen from the figure that water does not accumulate at any point. This idea is mathematically written as $\nabla\cdot\mathbf{V} = 0$. The fig.1.16 shows that water curls around the origin. It is clear from the figure that if a right handed screw is rotated along the direction of the flow of water, its tip advances along the negative z–direction.

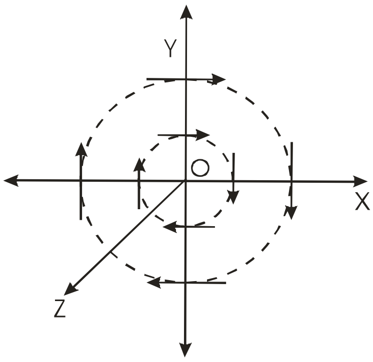


Fig.1.16: A schematic representation of the vector function y**i** – x**j**.

It may be noted that one <u>need not always physically see</u> a pure rotation of the vector fields when the curl is non-zero. From the expression of curl given above we see that curl of a vector is nonzero if $\frac{\partial B_z}{\partial y} \neq \frac{\partial B_y}{\partial z}$ or $\frac{\partial B_x}{\partial z} \neq \frac{\partial B_z}{\partial x}$ or $\frac{\partial B_y}{\partial x} \neq \frac{\partial B_x}{\partial y}$. For example, the vector **V** = y**i** has non zero curl. (Draw this vector function and calculate curl, and verify for yourself.). The vectors whose curl is zero are called **irrotational**.

Exercise:

20* Show that $\nabla \times \mathbf{F} = 0$ where $\mathbf{F} = -\frac{Gm_1m_2}{r^2}\hat{r}$

21. Show that **F** = yz **i** + zx **j** + xy **k** can be written as the gradient of a scalar function and as a curl of a vector function.

1.12 Three Fundamental Theorems in Vector Calculus

(1) Gauss's Theorem or Gauss Divergence Theorem or Divergence Theorem

The theorem states that the volume integral of the divergence of a vector function **A** taken over any volume is equal to the surface integral of **A** taken over the closed surface enclosing the volume. Mathematically, $\int_{volume} (\nabla.\mathbf{A})\, dv = \oint_{surface} \mathbf{A}.\mathbf{ds}$

Explanation using liquid flow:

Vector fields are often illustrated using the example of the velocity field of a liquid. The velocity, a speed and direction, of a moving fluid at each point can be represented by a vector **J**. Consider an imaginary closed surface *S* inside a liquid, enclosing a volume of liquid. The flux of liquid out of the volume is equal to the surface integral of the velocity over the surface, i.e., $\oint_{surface} \mathbf{J}.\mathbf{ds}$. (See sec.1.8(2)).

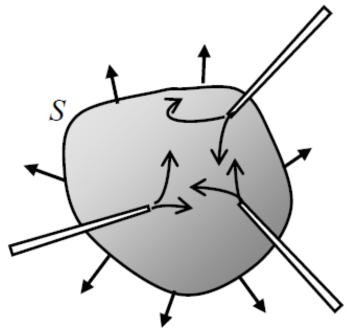
Since the amount of liquid inside a closed volume remains constant, if there are no sources or sinks inside the volume, the flux of liquid out of the surface imagined is zero, i.e., $\oint_{surface} \mathbf{J}.\mathbf{ds} = 0$. If the liquid is moving, it may flow into the volume at some points on the surface *S* and out of the volume at other points, but the amounts flowing in and out at any moment are equal, so the ***net*** flux of liquid out of the volume is zero.

However if a ***source*** of liquid is inside the closed surface, such as a pipe through which liquid is introduced (fig.1.17 shows three sources), the additional liquid will exert pressure on the surrounding liquid, causing an outward flow in different directions. This will cause a net outward flow through the surface *S*. The flux outward through *S* equals the volume rate of flow of fluid into *S* from the pipe. [Similarly if there is a sink or drain inside *S*, such as a pipe which drains the liquid off, the velocity throughout the liquid will be towards the location of the drain. The volume rate of flow of liquid inward through the surface *S* equals the rate of liquid removed by the sink.]

If there are multiple sources and sinks of liquid inside *S*, the flux through the surface can be calculated by adding up the volume rate of liquid added by the sources and subtracting the rate of liquid drained off by the sinks. The volume rate of flow of liquid through a source or sink (with the flow through a sink given a negative sign) is equal to the divergence of the velocity field at the pipe mouth. (The divergence of **V** is the net flux from unit volume, sec. 1.10). So adding up (i.e., integrating) the divergence of the liquid throughout the volume enclosed by *S*, $\int_{volume} \nabla.\mathbf{J}\,dv$, equals the volume rate of flux through the surface *S*. Hence, at steady state,

$$\int_{volume} \nabla.\mathbf{J}\,dv = \oint_{surface} \mathbf{J}.\mathbf{ds}$$ This is the divergence theorem.

Fig. 1.17: At equilibrium, the quantity of water comes in per second is equal to that flows out through the hypothetical surface *S* per second.



Note that the Gauss's theorem converts a volume integral into a surface integral and vice versa. Note also that the theorem says nothing about the shape of the surface. The only condition is that the volume considered for integration must be enclosed by the surface considered and the system must be in a steady state.

(2) Stokes' Theorem

Stokes' theorem states that the surface integral of the curl of a vector function **A** taken over any surface is equal to the line integral of **A** along the periphery of the surface.

$$\text{i.e.,} \quad \int_{\text{surface}} (\nabla \times \mathbf{A}).\mathrm{d}\mathbf{s} = \oint_{\text{line}} \mathbf{A}.\mathrm{d}\boldsymbol{l}$$

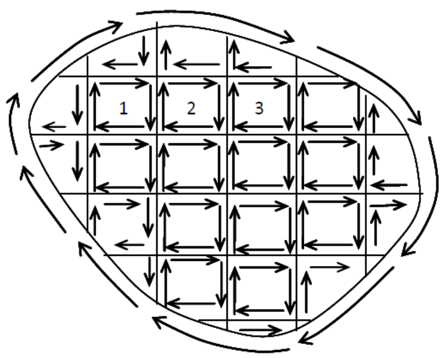


Fig.1.18: Figure shows certain vector field **A** over a bounded surface. Let the surface may be divided into many cells and $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$ etc. be the vectors at the cells 1,2,3, etc.. Then $\nabla\times\mathbf{A}_1$, $\nabla\times\mathbf{A}_2$, $\nabla\times\mathbf{A}_3$ etc. are the measure of the rotational effect of A at each cell. $\nabla\times\mathbf{A}_i.\mathrm{d}\mathbf{s}_i$ picks up the perpendicular component $\nabla\times\mathbf{A}_i$ at each surface element. Since the directions of the vector components responsible for the rotation are in opposite directions for the adjacent interior surface elements, only the components along the outer boundary survive on summation of $\nabla\times\mathbf{A}_i.\mathrm{d}\mathbf{s}_i$ over the surface..

Physically, Stokes' theorem means the following: Consider a vector function **A**, which has non–zero curl over a bound surface. $\nabla\times\mathbf{A}$ is a measure of the rotational effect (or circulation effect) of the vector field. The dot product '$\nabla\times\mathbf{A}.\mathbf{ds}$' picks up the components of $\nabla\times\mathbf{A}$ perpendicular to the surface element ds. Fig. 1.18 shows the rotational effect of certain vector function over a bounded surface. (Recall that in fig. 1.16, $\nabla\times\mathbf{A}$ is along –ve axis, and the vector **A** is in the x-y plane.) Addition of the normal component of $\nabla\times\mathbf{A}$ over the surface is equivalent to calculation of the net rotational effect of **A** parallel to the surface. The circulation effect of vector **A** over any cell lying inside the boundary cancels that due to the rotational effects of the adjacent cells. Only the components that survive after summation is the components of **A** lying along the boundary. (See fig. 1.18.) Thus, $\int_{\text{surface}}(\nabla\times\mathbf{A}).\mathbf{ds}$ is equivalent to $\oint_{\text{line}}\mathbf{A}.\mathrm{d}\boldsymbol{l}$.

Note: The surface can assume any shape: plane or curved or balloon shaped etc. Stokes' theorem converts a surface integral into a line integral, and vice versa. The Stokes' theorem also determines direction of the surface area vector for an open surface: if the fingers of our right hand point in the direction of line integral around the boundary, the outward stretched thump points along the direction of the surface area vector. Hence, in fig. 1.18, the surface area vector is into the plane of the paper.

Stokes' theorem suggests that for any vector **A**, $\int_{\text{surface}}(\nabla\times\mathbf{A}).\mathbf{ds}$ does not depend on the nature of the surface, but depends only on the boundary of the surface. Consider a balloon shaped surface. $\int_{\text{surface}}(\nabla\times\mathbf{A}).\mathbf{ds}$ is equal to $\oint_{\text{line}}\mathbf{A}.\mathrm{d}\boldsymbol{l}$ along the mouth of the balloon. When the mouth of the balloon shrinks to a point, $\oint_{\text{line}}\mathbf{A}.\mathrm{d}\boldsymbol{l}$ becomes zero, and the surface becomes a closed surface. Hence $\int_{\text{surface}}(\nabla\times\mathbf{A}).\mathbf{ds}$ becomes $\oint_{\text{surface}}\nabla\times\mathbf{A}.\mathbf{ds}$, and is zero according to Stokes' theorem. That is $\oint_{\text{surface}}(\nabla\times\mathbf{A}).\mathbf{ds}=0$ for any vector **A.**

Gauss's divergence theorem gives, $\oint_{\text{surface}} (\nabla \times \mathbf{A}).d\mathbf{s} = \int_{\text{volume}} \nabla.(\nabla \times \mathbf{A})\, dv$. This is valid for any volume considered. This implies that, for a **any vector A**, **∇.(∇×A) = 0**. If we denote ∇×**A** = **B**, we get ∇.**B** = 0. That is **B** is **solenoidal**. In other words, any solenoidal vectors (here vector **B**) can always be represented as the curl of a vector (here vector **A**). The vector whose curl gives a solenoidal vector is called the **vector potential** associated with the solenoidal vector. In this example, **A** is the vector potential of **B**.

(3) Theorem for Gradient

If φ is a scalar function then, $\int_A^B \nabla\varphi.d\boldsymbol{l} = \varphi(B) - \varphi(A)$

Explanation of the gradient theorem: Let φ be a scalar function. Then ∇φ is a vector function, which gives the maximum rate of change in the scalar function for infinitesimal change in the position (see section 1.9). We want to find the net change in φ as we go along a curve from point A to point B. For this, divide the curve AB into a large number of small segments having length d*l*.

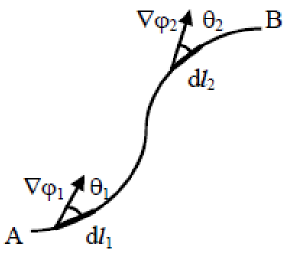


Fig.1.19: Figure shows that ∇φ $\cos\theta_1$ is the component of ∇φ along the segment $d\boldsymbol{l}_1$ and ∇φ $\cos\theta_2$ is the component of ∇φ along the segment $d\boldsymbol{l}_2$.

From fig.1.19, $\nabla\varphi.d\boldsymbol{l}_1 = \nabla\varphi\, dl_1 \cos\theta_1$. Here, $\nabla\varphi\cos\theta_1$ is the component of ∇φ along the segment $d\boldsymbol{l}_1$. Similarly $\nabla\varphi.d\boldsymbol{l}_2 = \nabla\varphi\, dl_2\cos\theta_2$, and $\nabla\varphi\cos\theta_2$ is the component of ∇φ along the segment $d\boldsymbol{l}_2$. From eqn.3 in section 1.9, (∇φcosθ) (d*l*) = Δφ, which is the change in φ as one moves along d***l***. Thus, if there are n segments, the total change occurs along the entire curve AB $= \sum_{i=1}^{n} \Delta\varphi_i = \sum_{i=1}^{n} \nabla\varphi_i . d\boldsymbol{l}_i$

If the line segments are infinitesimally small, the summation becomes an integration and we get the total change in φ as $\sum_{i=1,n\to\infty}^{n} \nabla\varphi_i.d\boldsymbol{l}_i = \int_A^B \nabla\varphi.d\boldsymbol{l}$. But, the change in the value of φ = The value of φ at B − The value of φ at A.

That is, $\int_A^B \nabla\varphi.d\boldsymbol{l} = \varphi(B) - \varphi(A)$, where φ(B) is the value of the function at B and φ(A) is that at A. This is the physical interpretation of the gradient theorem.

If ∇φ = **V**, $\int_A^B \mathbf{V}.d\boldsymbol{l} = \varphi(B) - \varphi(A)$.

Then, for any vector function **V** given by the gradient of a scalar function φ, the line integral of **V** along any line segment is equal to the difference in the values of the scalar function φ at the end points. That is, $\int_A^B \mathbf{V}.dl$ is independent of the path, if V=∇φ. For a closed loop $\oint_{line} \mathbf{V}.d\boldsymbol{l} = \varphi(A) - \varphi(A) = 0$. In other words, if $\oint_{line} \mathbf{V}.d\boldsymbol{l} = 0$ for some vector function **V**, one can always find a scalar function φ such that **V** = ∇φ. In physics, we call this φ as the **potential function** of the vector field **V**. In other words, $\oint_{line} \nabla\varphi.d\boldsymbol{l} = 0$. Using Stokes' theorem, $\oint_{line} \nabla\varphi.d\boldsymbol{l} = \int_{surface} (\nabla\times(\nabla\varphi)).d\mathbf{s} = 0$, for any shape of the surface. This implies that **for any scalar function** φ, ∇×∇φ = 0 always.

1.13 Some Important Vector Identities

The following vector identities regarding vector functions **A**, **B** and **C** and scalar functions 'f' and 'g' are very important, and are often used in this book. Proofs of these identities are not worth doing here. These identities can be verified using exercises 23 and 24. The operator ∇**.**∇, called the 'Laplace operator' or 'Laplacian', is denoted as ∇^2 in the following equations.

In Cartesian coordinate system $\nabla^2 = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial z^2}$. For a scalar function f, $(\nabla.\nabla)f$ is denoted as $\nabla^2 f$. For a vector function **A,** $\nabla^2\mathbf{A}$ implies $\nabla^2 A_x\mathbf{i} + \nabla^2 A_y\mathbf{j} + \nabla^2 A_z\mathbf{k}$.

(1) $\nabla.(\nabla\times\mathbf{A}) = 0$

(2) $\nabla\times(\nabla f) = 0$

(3) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla.\mathbf{A}) - \nabla^2\mathbf{A}$

(4) $\nabla.(f\mathbf{A}) = f(\nabla.\mathbf{A}) + \mathbf{A}.(\nabla f)$

(5) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(6) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(7) $\nabla.(\mathbf{A}\times\mathbf{B}) = \mathbf{B}.(\nabla\times\mathbf{A}) - \mathbf{A}.(\nabla\times\mathbf{B})$

(8) $\nabla(\mathbf{A}.\mathbf{B}) = \mathbf{A}\times(\nabla\times\mathbf{B}) + (\mathbf{A}.\nabla)\mathbf{B} + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{B}.\nabla)\mathbf{A}$

Exercise:

22* If the charge enclosed by a spherical surface decreases at the rate of 4 C/s, find the total electric current through the surface. Find also the average current density through the surface.

23. Let $A = 3x^2\mathbf{i} - z^2\mathbf{j} - y\mathbf{k}$ and $B = y^2\mathbf{i} + 6\mathbf{k}$. Show that

(a) $\nabla.(\nabla\times\mathbf{A}) = 0$; (b) $\nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla.\mathbf{A}) - \nabla^2\mathbf{A}$;

(c) $\nabla.(\mathbf{A}\times\mathbf{B}) = B.(\nabla\times\mathbf{A}) - \mathbf{A}.(\nabla\times\mathbf{B})$

24. Let $g = x^2 - 3z^2$, $f = 3y - x^2$ and $\mathbf{A} = 3y^2\mathbf{i} - z^2\mathbf{j}$. Show that

(a) $\nabla\times(f\mathbf{A}) = f(\nabla\times\mathbf{A}) - \mathbf{A}\times(\nabla f)$

(b) $\nabla(fg) = f(\nabla g) + g(\nabla f)$

(c) $\nabla.(f\mathbf{A}) = f(\nabla.\mathbf{A}) + \mathbf{A}.(\nabla f)$

(d) $\nabla\times(\nabla f) = 0$.

25* Show that the vector area **A** bounded by a loop in a plane is $\mathbf{A} = \frac{1}{2}\oint_{loop}\mathbf{r}\times d\boldsymbol{l}$

26. Show that $\nabla.\frac{\hat{r}}{r^2} = 0$ if $r \neq 0$ and $\nabla.\frac{\hat{r}}{r^2} = 4\pi$ if $r = 0$.

1.14 Orthogonal Co-ordinate Systems

In order to study the physical interactions happening in a three dimensional space, we need a coordinate system that span the entire 3D space. There are a variety of 3D coordinates systems. Many problems in physics are highly simplified when we write the expressions in a suitable coordinate system, instead of conventional Cartesian coordinate system. Three important orthogonal co-ordinate systems widely used are briefly described in this section. They are

1. Rectangular or Cartesian co-ordinate system
2. Spherical polar co-ordinate system
3. Cylindrical co-ordinate system

1. Rectangular Cartesian Co-ordinate System

It is assumed that the readers are familiar with this system. We shall restrict ourselves to the right-handed co-ordinate system. In this system, the co-ordinate of any point in space are taken as the distances from the origin along three mutually perpendicular axes, and written as (x,y,z). The unit vectors along x,y and z axes are respectively denoted as **i**, **j** and **k.** These unit vectors are mutually perpendicular. Let the co-ordinate of a point A be (3,6,–2). Then 3**i** + 6**j** –2**k** is the position vector of the point A.

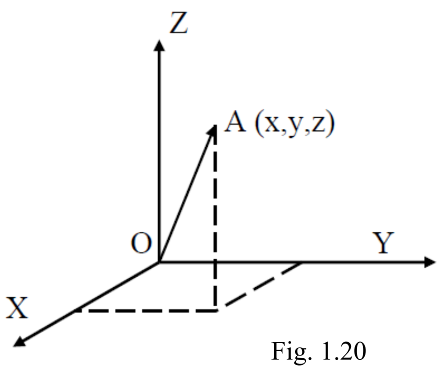


Fig. 1.20

The following points are worth noting in this coordinate system.

- The separation vector drawn from D (x_1,y_1,z_1) to a point C (x_2,y_2,z_2) is given by (x_2-x_1)**i** + (y_2-y_1)**j**+(z_2-z_1)**k**.
- An infinitesimal displacement in this system is d***l*** = dx**i** + dy**j** + dz**k**
- A volume element in this system is dv = (dx)(dy)(dz)
- An elemental area perpendicular to z-axis is (dx)(dy)**k**
- An elemental area perpendicular to x axis is (dy)(dz)**i**